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ORIGINAL PAPER

Chaos of a coupled lattice system related with the Belusov–Zhabotinskii reaction

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Abstract In this paper we present a lattice dynamical system stated by Kaneko in (Phys Rev Lett, 65: 1391–1394, 1990) which is related to the Belusov–Zhabotinskii chemical reaction. We prove that this CML (Coupled Map Lattice) system is chaotic in the sense of Li–Yorke and in the sense of Devaney for zero coupling constant. Some problems on the dynamics of this system are stated for the case of having non-zero coupling constant.

Keywords Coupled map lattice · Chaos in the sence of Li–Yorke · Chaos in the sence of Devaney

1 Introduction

Classical discrete dynamical systems (DDS's), i.e., a couple composed by a space X (usually compact and metric) and a continuous self-map ψ on X, have been highly

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considered in the literature (see e.g., [3] or [8]) because are good examples of problems coming from the theory of Topological Dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences (see for example, [7,16,19] or [18]). In most cases in the formulation of such models ψ is a C^{∞}, an analytical or a polynomial map.

Coming from physical/chemical engineering applications, such a digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a generalization of DDS's have recently appeared as an important subject for investigation, we mean the so called *Lattice Dynamical Systems* or *1d Spatiotemporal Discrete Systems*. In the next section we provide all the definitions. To show the importance of these type of systems, see for instance [5].

To analyze when one of this type of systems have a complicated dynamics or not by the observation of one topological dynamics property is an open problem. The aim of the present paper is, by using the notion of *chaos*, to characterize the dynamical complexity of a coupled lattice system stated by Kaneko in [14] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction. We present some other problems for the future related with physical/chemical applications.

2 Notation and basic construction

Let us start introducing two of the most well-known notions of chaos.

Definition 1 A pair of points $x, y \in X$ is called a *Li-Yorke pair* if

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A set $\mathcal{S} \subset X$ is called a *LY-scrambled set* for f (*Li-Yorke set*) if $\#\mathcal{S} \geq 2$ and every pair of different points in $\mathcal{S}$ is a LY-pair where # means the cardinality.

For continuous self-maps on the interval [0,1], Li and Yorke [17] suggested that a map should be called "chaotic" if it admits an uncountable scrambled set. This was subsequently accepted as a formal definition.

Definition 2 We say that a map f is *Li and Yorke chaotic* if it has an uncountable LY-scrambled set.

One may consider weaker variants of chaos in the sense of Li and Yorke based on the cardinality of scrambled sets (see for instance [10]).

On the other hand, a map f is:

(1) *transitive* if for any pair of nonempty open sets $\mathcal{U}, \mathcal{V} \subset X$ there exists an $n \in \mathbb{N}$ such that $f^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$;
(2) *locally eventually onto* if for every nonempty open set $\mathcal{U} \subset X$ there exists an $m \in \mathbb{N}$ such that $f^m(\mathcal{U}) = X$. Since this property can be regarded as the topological analog of exactness defined in ergodic theory, it is often called *topological exactness*. We use the second name here.

Recall that a periodic point of period n of f is a point x such that $f^n(x) = x$ and $f^j(x) \neq x$ for $0 < j < n$.

Definition 3 A map f is called *Devaney* chaotic if it satisfies the following two properties:

(1) f is transitive,
(2) the set of periodic points of f is dense in X.

The original definition given by Devaney [8] contained an additional condition on f, which reflects unpredictability of chaotic systems: *sensitive dependence on initial conditions*. However, it was proved see, e.g., [2] that sensitivity is a consequence of transitivity and dense periodicity under the assumption that X is an infinite set.

2.1 Lattice dynamical systems

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \left\{ x : x = \{x_i\},\ x_i \in \mathbb{R}^d,\ i \in \mathbb{Z}^D,\ \|x_i\| < \infty \right\},$$

where $d \geq 1$ is the dimension of the range space of the map of state x_i, $D \geq 1$ is the dimension of the lattice and the l^2 norm $\|x\|_2 = \left(\Sigma_{i \in \mathbb{Z}^D} \mid x_i \mid^2\right)^{1/2}$ is usually taken ($\mid x_i \mid$ is the length of the vector x_i).

We deal with the following 1d-LD CML (Coupled Map Lattice) system which was stated by Kaneko in [14] (for more details see for references therein) and it is related to the Belusov–Zhabotinskii reaction (see [16] and for experimental study of chemical turbulence by this method [11–13]):

$$x_n^{m+1} = (1 - \epsilon) f\left(x_n^m\right) + \epsilon/2 \left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right) \right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots L$), ϵ is coupling constant and $f(x)$ is the *unimodal map* on the unite closed interval $I = [0, 1]$, i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ such that $f(c) = 1$. For simplicity we will deal with so called "tent map", defined by

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2), \\ 2 - 2x, & x \in [1/2, 1]. \end{cases} \tag{2}$$

In general, one of the following periodic boundary conditions of the system (1) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

The Eq. (1) was studied by many authors, mostly experimentally or semi-analytically than analytically. The first paper with analytic results is [6], where it was proved that this system is Li–Yorke chaotic, we give alternative and easier proof of it in this paper.

We consider, as an example the 2-element one-way coupled logistic lattice (OCLL, see [15]) $H : I^2 \to I^2$ written as

$$x_1^{m+1} = (1-\epsilon) f\left(x_1^m\right) + \epsilon f\left(x_2^m\right),$$
$$x_2^{m+1} = \epsilon f\left(x_1^m\right) + (1-\epsilon) f\left(x_2^m\right), \tag{3}$$

where f is the tent map.

3 Main results

The following two lemmas will be used for the proof of the main results. The proof of the first one is obvious (or, see e.g., [9]).

Lemma 1 *Let $f : X \to X$ and $g : Y \to Y$ be maps with dense sets of periodic points. Then the Cartesian product $f \times g : X \times Y \to X \times Y$ has also dense set of periodic points.*

Proposition 1 ([3]) *Let f be the tent map defined by* (2). *Put $I_{k,l} = [(l-1)/2^k, l/2^k]$ where $l = \{1, 2, 3, \ldots, 2^k\}$ and $k \in \mathbb{N}$. Then the restriction of f^k to $I_{k,l}$ is linear homeomorphism onto [0,1].*

Let us note that the Cartesian product of two topologically transitive maps is not necessarily topologically transitive (see e.g., [9]). Hence, for the proof of Theorem 4 we need to prove:

Lemma 2 *The system*

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right],$$

is topologically exact for $\epsilon = 0$.

Proof Let U be given open subset of I^L. Then the projection of U to the mth coordinate contains U_m open connected subset of I, for each $m = 1, 2, \ldots L$. Then by Proposition 1 there is k_m such that $f^{k_m}(U_m) = I$. If we put $K = \max\{k_m | m = 1, 2, \ldots L\}$ then the Kth iteration of U by the system (1) equals to I^L. □

Theorem 4 *The system*

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right],$$

is chaotic in the sense of Devaney for $\epsilon = 0$.

Proof The assertion follows by Lemma 1 and 2. □

The following Proposition is very powerful tool of symbolic dynamics[1] for observing nearly all dynamical properties.

Proposition 2 ([10]) *There is a subsystem of* (1) *which is conjugated*[2] *to* $\left(\Sigma_2^L, \sigma_2^L\right)$.

Theorem 5 *The system*

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right],$$

is chaotic in the sense of Li–Yorke for $\epsilon = 0$.

Proof By Proposition 2 the system (1) has a subsystem conjugated to (Σ_2^L, σ_2^L) which is Li–Yorke chaotic (see e.g., [4]). □

4 Concluding remarks

There are many other notions of chaos, like distributional-chaos, ω-chaos or to satisfy the specification property. The system (1) fulfils all this chaotic behaviour by the same arguments as in the proof of the Theorem 5. But obviously this system is not minimal, where minimal means that there is no proper subset which is invariant, nonempty and closed.

For non-zero coupling constants the dynamical behaviour of the system (1) is more complicated. The first question is how the invariant subsets of phase space look like? Secondly, what are the properties of ω-limit sets (i.e., set of limits points of the trajectories)? The answer for these questions will be nontrivial. Similar system was studied in [1] and there was used the method of resultants to prove existence of periodic points of higher order. The same concept like in [1] should be used.

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[1] Here, σ_2 is the shift operator on the space of all two element sequences Σ_2.

[2] We say that two dynamical systems (X, f) and (Y, g) are *topologically conjugated* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, such homeomorphism is called *conjugacy*.

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